THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Suggested Solutions for HW2

1. If S is a non-empty subset of R which is bounded from above but not below. Suppose the following holds: $[x, y] \subset S$, for all $x, y \in S$. Show that S is either $(-\infty, \alpha]$ or $(-\infty, \alpha)$ for some $\alpha \in \mathbb{R}$.

   Solution. By the completeness of $\mathbb{R}$, we know that $\sup S$ exists. Let $x \in S$. Then clearly we have $x \leq \sup S$, so $S \subset (-\infty, \sup S]$. It remains to show that $(-\infty, \sup S) \subset S$, in which case either $S = (-\infty, \sup S)$ or $S = (-\infty, S]$. Suppose for the sake of contradiction that there is a $y \in (-\infty, \sup S)$ but $y \notin S$. Since $y \leq \sup S$, we can find a $x_1 \in S$ such that $y < x_1 \leq \sup S$. Also, since S is not bounded from below, we can likewise find an $x_2 \in S$ such that $x_2 < y$. Then since both $x_1, x_2 \in S$, by assumption we have that $[x_2, x_1] \subset S$ but this contradicts the fact that $y \notin S$. Hence we must have that $y \in S$. So we see that S is either $(-\infty, \alpha)$ or $(-\infty, \alpha]$ where $\alpha = \sup S$. ◀

2. Using $\varepsilon - N$ terminology, show the followings:

   (a) $\displaystyle\lim_{n\to\infty} \frac{n}{n^2-2} = 0.$

   (b) $\displaystyle\lim_{n\to\infty} (2n)^{\frac{1}{n}} = 0.$

   (c) $\displaystyle\lim_{n\to\infty} \frac{2^n}{n!} = 0.$

   Solution. (a) Let $\varepsilon > 0$ be given. Note that for $n \geq 2$, $n^2 - 2 \leq n^2 - n$ and so we have
   $$\left|\frac{n}{n^2-2}\right| \leq \left|\frac{n}{n^2-n}\right| = \left|\frac{1}{n-1}\right|.$$
   So taking $N > \frac{1}{\varepsilon} + 1$, we have for all $n \geq N$
   $$\left|\frac{n}{n^2-2}\right| \leq \left|\frac{1}{n-1}\right| < \frac{1}{\frac{1}{\varepsilon}+1-1} = \varepsilon$$
   as required.

   (b)

   (c) Let $\varepsilon > 0$ be given. For $n > \frac{1}{2}$, we note that $(2n)^{\frac{1}{n}} > 1$, so for each n we can write $(2n)^{\frac{1}{n}} = 1 + k_n$ for a sequence of positive numbers k_n. Then, if we can show that $k_n \to 0$ as $n \to \infty$, then we would have
   $$\left|(2n)^{\frac{1}{n}} - 1\right| = |1 + k_n - 1| = |k_n| < \varepsilon$$

for $n \geq K(\varepsilon)$ for some $K(\varepsilon) \in \mathbb{N}$, and we would be done.

Since we have $(2n)^{\frac{1}{n}} = 1 + k_n \implies 2n = (1+k_n)^n$, we use the binomial theorem to write

$$2n = (1+k_n)^n = 1 + nk_n + 2n(n-1)k_n^2 + \cdots \geq 1 + nk_n + 2n(n-1)k_n^2.$$

Re-arranging, we have

$$k_n^2 \leq \frac{2(2n-1)}{n(n-1)} \leq \frac{4n}{n(n-1)} = \frac{4}{n-1}.$$

So choosing $K(\varepsilon) > \frac{4}{\varepsilon} + 1$, we are done.

(d) Let $\varepsilon > 0$ be given. Note that for $n \geq 2$, we have

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot \cdots \cdot 2 \cdot 2}{1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n} \leq \frac{4}{n}.$$

So we have that for $N > \frac{4}{\varepsilon}$, we have for all $n \geq N$

$$\left|\frac{2^n}{n!}\right| \leq \frac{4}{n} < \varepsilon$$

as required.

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3. Suppose (x_n) is a sequence of positive real number such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L \in \mathbb{R}$.

   (a) Show that (x_n) is convergent if $L \in [0,1)$.

   (b) Can we conclude the convergence if $L = 1$? Justify your answer.

Solution. (a) We will show that when $L \in [0,1)$, (x_n) converges to 0. Let $\varepsilon > 0$ be given. Then since $\frac{x_n + 1}{x_n}$ converges to L, we know that there is an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$0 < \frac{x_{n+1}}{x_n} - L < \varepsilon \Leftrightarrow 0 < \frac{x_{n+1}}{x_n} < L + \varepsilon.$$

Since $L < 1$, $\alpha := L + \varepsilon_0 < 1$ for ε_0 chosen to be sufficiently small. Then for $n \geq N$, we have that

$$0 < x_n = x_N \cdot \frac{x_{N+1}}{x_N} \cdot \cdots \cdot \frac{x_n}{x_{n-1}} < x_N \alpha^{n-N+1}.$$

Since $0 < \alpha < 1$, the right hand side converges to 0 as $n \to \infty$ and we conclude that x_n converges to 0 by the squeeze theorem.

(b) No we cannot conclude the convergence when $L = 1$. Consider the sequence $x_n = n$. Then we have that

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

which converges to 1. But clearly x_n is unbounded from above and hence not convergent.

◀

4. If $x_1 = \sqrt{2}$ and

$$x_{n+1} = \sqrt{2 + \sqrt{x_n}}$$

for all $n \in \mathbb{N}$. Show that (x_n) is convergent and that $x_n < 2$ for all $n \in \mathbb{N}$.

Solution. We first show the upper bound by mathematical induction. Clearly $x_1 = \sqrt{2} < 2$. Suppose for our inductive hypothesis that $x_k < 2$ for some $k \in \mathbb{N}$. Then we have

$$x_{k+1} = \sqrt{2 + \sqrt{x_k}} < \sqrt{2 + \sqrt{2}} < \sqrt{2+2} = 2.$$

So (x_n) is bounded from above by 2.

We next show that x_n is increasing. We have that $x_2 = \sqrt{2 + \sqrt{1}} = \sqrt{3} > \sqrt{2} = x_1$, so the base case is satisfied. Suppose for our inductive hypothesis that $x_k \leq x_{k+1}$ for some $k \in \mathbb{N}$. Then we have

$$x_{k+2} = \sqrt{2 + \sqrt{x_{k+1}}} \geq \sqrt{2 + \sqrt{x_k}} = x_{k+1}$$

since the square root function is increasing. So x_n is increasing.

Then we can conclude that (x_n) converges by the Monotone Convergence Theorem.

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5. If $x_n = \sum_{k=1}^{n} a_k$ for some sequence (a_k). Suppose (x_k) is convergent, and (b_k) is another sequence of positive real number which is monotonic increasing and bounded, show that $y_n = \sum_{k=1}^{n} a_k b_k$ is convergent.

Solution. Let $\varepsilon > 0$ be given. First note that (b_k) converges by the Monotone Convergence Theorem to some limit, say $b \in \mathbb{R}$. Moreover, we have that $b_k \leq b$ for all $k \in \mathbb{N}$. Let $a := \lim_{n\to\infty} x_n$. We want to show that y_n converges to ab. We have

$$\begin{aligned}
\left|\sum_{k=1}^{n} a_k b_k - ab\right| &= \left|\sum_{k=1}^{n} a_k b_k - ab_n + ab_n - ab\right| \\
&\le \left|\sum_{k=1}^{n} a_k b_k - ab_n\right| + |ab_n - ab| \\
&\le \left|\sum_{k=1}^{n} a_k b_n - ab_n\right| + |a|\,|b_n - b| \\
&\le |b_n|\left|\sum_{k=1}^{n} a_k - a\right| + |a|\,|b_n - b| \\
&\le b\left|\sum_{k=1}^{n} a_k - a\right| + |a|\,|b_n - b|
\end{aligned}$$

where in the third inequality we used the fact that b_k is monotonically increasing and in the last inequality we used the fact that $b_k \le b$ for all $k \in \mathbb{N}$. Since x_n converges to a, there is an $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, we have

$$\left|\sum_{k=1}^{n} a_k - a\right| < \frac{\varepsilon}{2b}$$

and since b_n converges to b, there is an $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, we have

$$|b_n - b| < \frac{\varepsilon}{2a}.$$

Then taking $N = \max\{N_1, N_2\}$ yields the desired result. ◀